Vectors – Formulas and Tutorials

Reviewed by frn

Formulas

- Addition of vectors: $\begin{pmatrix} a_x \\ a_y \end{pmatrix} + \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix}$
- Scalar multiplication: $n * \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} n * a_x \\ n * a_y \end{pmatrix}$
- Magnitude of a vector: $\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$
- Unit vector : $\overrightarrow{a_E} = \frac{1}{|\vec{a}|} * \vec{a}$
- Dot product: $\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x * b_x + a_y * b_y + a_z * b_z$ OR $dotP(\vec{a}, \vec{b})$
- Angle between two vectors: $\cos(\alpha) = \frac{\vec{a} * \vec{b}}{|\vec{a}| * |\vec{b}|}$
- Line: $\vec{x} = \vec{a} + t * \overrightarrow{AB}$
- Plane: vector equation $E: \vec{x} = \vec{a} + t * \vec{u} + s * \vec{v}$
- Plane: normal vector equation $E: (\vec{x} - \vec{p}) * \vec{n} = 0$
- Plane: Cartesian equation $E: a * x + b * y + c * z = d$
- Hessian normal form: $E: (\vec{x} - \vec{p}) \cdot \overrightarrow{n_E} = 0$
- Distance between point R and plane: $|(\vec{r} - \vec{p}) \cdot \overrightarrow{n_E}|$ OR $\left|\frac{a \cdot r_1 + b \cdot r_2 + c \cdot r_3 - d}{\sqrt{a^2 + b^2 + c^2}}\right|$
- Cross product: $\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$ OR $crossP(\vec{a}, \vec{b})$
- Circle: $(x_1 - m_1)^2 + (x_2 - m_2)^2 = r^2$
- Sphere $(x_1 - m_1)^2 + (x_2 - m_2)^2 + (x_3 - m_3) = r^2$
- Circle or sphere: vector equation $(\vec{x} - \vec{m})^2 = r^2$ (for squaring, use the dot product)

Good to Know

- If the dot product is zero, the two vectors are perpendicular
- If the cross product is zero, the two vectors are parallel
- Area of a triangle is $\frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2}$
- A vector perpendicular to two other vectors can be calculated using $\vec{n} = \vec{a} \times \vec{b}$, $\overrightarrow{n_E} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

Tutorials

See document for the previous test!

Distance between line and point

- First, we will determine a general point P_t on the line.
- Then, we will calculate the dot product of $\overrightarrow{RP_t}$ and the direction vector $\vec{u}$ of the line. As the distance is minimal when $\overrightarrow{RP_t} \perp \vec{u}$, the dot product has to be zero. We solve for t.
- Then, we plug t into the vector equation of the line and calculate the coordinates of the point P_t and calculate the distance between R and P.

1. Any point P_t on the line has the coordinates $P_t(1 + 2t|1 + 3t|t)$.

2. $\overrightarrow{RP_t} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} 2t \\ -1 + 3t \\ t - 0.08 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0 \Leftrightarrow 4t - 3 + 9t + t - 0.08 = 0 \Leftrightarrow t = 0.22$

3. $|\overrightarrow{RP}_{0.22}| = \left| \begin{pmatrix} 2 \cdot 0.22 \\ -1 + 3 \cdot 0.22 \\ 0.22 - 0.08 \end{pmatrix} \right| = \sqrt{0.3288} \approx 0.573$

The plane is approximately 573 metres away at its closest distance to you.

You can also find the distance between a point R and a line g if you first calculate the general distance between P_t and R, and then minimize this distance (you can use DC to minimize).

Distance between two skew lines

1. Given are $g: \vec{x} = \vec{p} + t \cdot \vec{u}$ and $h: \vec{x} = \vec{q} + t * \vec{v}$
2. Calculate $\overrightarrow{n_E}, \overrightarrow{n_E} \perp \vec{u}, \vec{v}$
3. $d = |(\vec{p} - \vec{q}) \cdot \overrightarrow{n_E}|$

Tangents

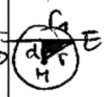
Point B on the circle

1. Vector $\overrightarrow{MB}$
2. Find $\vec{n}, \vec{n} \perp \overrightarrow{MB}$
3. Tangent line: $\vec{x} = \vec{B} + t \cdot \vec{n}$

Point P outside of the circle

1. Calculate midpoint $M_{MP} = \frac{\overrightarrow{MP}}{2}$
2. Circle with center M_{MP} and radius $\frac{M_{MP}}{2}$
3. Intersection points of the two circles are boundary points of the tangents
4. Lines through P and $B_{1,2}$

Position relations

	Objects intersect if…	Object of intersection	Calculations needed to determine the position relation / object of intersection
Point & Sphere P M, r	(A) distance between P and $M = r$ (B) the eq. of the sphere is true for P	Point	(A) find $\lvert\overrightarrow{MP}\rvert$ (B) insert p_1, p_2, p_3 into the equation of the sphere for x_1, x_2 and x_3
Line & Sphere g	(A) distance between M and $g < r$ (B) there is a solution to the sys. of eq.	Point(s)	(A) find $dist(M, g)$ (see ch. 5C) (B) Insert general pt. P_t into the eq. of the sphere, solve for t.
Plane & Sphere E	… distance d between M and E is less than r.	Circle M_1, r_1	Radius: Use Pythagoras: →  $r_1 = \sqrt{r^2 - d^2}$ Centre: $\vec{M_1} = \vec{M} + \vec{n_E} \cdot d$ (choose $\vec{n_E}$, such that $M_1 \in E$!)
	$d = r$		Tangent plane to a sphere: (through $B \in$ sphere) $\vec{n} = \overrightarrow{MB}$ $\Rightarrow$ normal vector eq: $(\vec{x} - \vec{B}) \cdot \vec{n} = 0$
Sphere & Sphere M_1 M_2 r_1 r_2	$\lvert\overrightarrow{M_1M_2}\rvert > \lvert r_1 - r_2\rvert$ and $\lvert\overrightarrow{M_1M_2}\rvert < r_1 + r_2$	Circle M^* r^*	Radius: Use Pyth. 2x & syst. of eq. Centre: $\overrightarrow{M^*} = \vec{M_1} + \frac{\overrightarrow{M_1M_2}}{\lvert\overrightarrow{M_1M_2}\rvert} \cdot a$