Lecture Summary

Note Some of the contents were typed during the in-class summary at the end of the term (thus in English), some parts are copied from the scan of the in-class summary by Prof. Imamoglu and other contents (mostly of the "nice to know/techniques" section) are from the internet (preferably .edu sites).

Contents

Kapitel 1 Einführung

- **Prinzip der Induktion** Sei für jedes $n \in \mathbb{N}, A(n)$ eine Behauptung geeben. Soll die Behauptung $A(n)$ für alle $n \in \mathbb{N}$ bewiesenw erden, so genügen dazu zwei Beweisschritte:
  - 1. Der Beweis von $A(0)$ $\big(A(m)\big)$
  - 2. für jedes $n \geq 0$ $(n \geq m)$: $A(n) \Rightarrow A(n+1)$
- **Indirekter Beweis** Wenn wir die Aussage $A \Rightarrow B$ beweisen möchten, fügen wir $\neg B$ als Annahme hinzu und nach eineer Kette von erlaubten Schlüssen kommen wir zu einer flaschen Aussage. $(A \Rightarrow B) \equiv (\neg B \Rightarrow A)$
- Surjektivität:

$$f: x \to y$$
$$\forall y \in Y \exists x \in X, sd\ f(x) = y$$

- Injektivität:

$$f: x \to y$$
$$(f(x_1) = f \to x_1 = x_2)$$

- Bijektivität: Surjektivität. & Injektivität

Kapitel 2 Die Reellen Zahlen

- $(\mathbb{R}, +, \cdot)$ ist ein Körper
- **Körperaxiome** $(R, +)$ Abelische Gruppe: $\forall x, y, z \in \mathbb{R}$
  - A1 $x + y = y + x$
  - A2 $x + (y + z) = (x + y) + z$
  - A3 $x + 0 = 0 + x = x$
  - A4 $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}$ mit $x + y = 0$
  - M1 $x \cdot y = y \cdot x$
  - M2 $x(yz) = (xy)z$
  - M3 $x \cdot 1 = 1 \cdot x = x$
  - M4 $x \neq 0, \exists y \in \mathbb{R}$ mit $xy = 1 = yx$
  - D $x(y + z) = xy + xz$
- **Ordnungsaxiome** $\forall x, y, z \in \mathbb{R}$
  - O1 $x \leq x$
  - O2 $x \leq y$ und $y \leq z \Rightarrow x \leq z$
  - O3 $x \leq y$ und $y \leq x \Rightarrow x = y$
  - O4 entweder $x \leq y$ oder $y \leq x$
  - OA $x \leq y \Rightarrow x + z \leq y + z$
  - OM $x, y \geq 0 \Rightarrow xy \geq 0$
- **Ordunungsvollständigkeitsaxiom** Seien $A, B \subset \mathbb{R}$ nicht leere Teilmengen, sodass $a < b\ \forall a \in A, b \in B$. Dann gibt es $c \in \mathbb{R}$ mit $a \leq c \leq b\ \forall a \in A, b \in B$

- **Definition** $|x| = \max\{x, -x\}$
- **Dreiecksungleichung**
  - $|x + y| \leq |x| + |y|$
  - $|xy| \leq |x||y|$
- **Satz 2.10** Every bounded from above set in $\mathbb{R}$ has a Supremum, every bounded from below set in $\mathbb{R}$ has an Infimum
- **Corollary 2.11**
  - $E \subset F, F$ bounded above: $\sup E \leq \sup F$
  - $E \subset F, F$ bounded below: $\inf F \leq \inf E$
  - If E has a Supremum then $\exists x \in E, x > \sup E - \delta$
  - If E has an Infimum then $\exists x \in E\ , x < \inf E + \delta$
- **Satz 2.13 Archimedische Eigenschaft** Zu jeder Zahl $0 < b \in \mathbb{R}$ gibt es ein $n \in \mathbb{N}$ mit $b < n$
- **Korollar 2.14**
  - 1. Seien $x > 0$ und $y \in \mathbb{R}$ gegeben, Dann gibt es $n \in \mathbb{Z}$ mit $y < nx$
  - 2. $\forall x, y, a \in \mathbb{R}$ die die Ungleichung $a \leq x \leq a + \frac{y}{n}\ \forall n \in \mathbb{N}$ erfüllen, ist $x = a$
- **Eukildische Räume** $(\mathbb{R}^n, +, -)$ componenentwise addition; scalar multiplication $\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$, $(\mathbb{R}^n, +, \cdot)$ is a vector space,
- **Skalarprodukt** $\langle x, y\rangle := x_1 y_1 + \dots + x_n y_n$
  - SP1 $\langle x, y\rangle = \langle y, x\rangle$
  - SP2 $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$
  - SP3 $\langle x, x\rangle \geq 0,\ \langle x, x\rangle = 0 \Leftrightarrow x = 0$
- **Satz 2.19 Cauchy-Schwarz** $\|x + y\| \leq \|x\| + \|y\|,\ \|x\| := \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^n x_i^2}$
- **Satz 2.20** $\|\alpha x\| = |\alpha|\|x\|$ und $\|x + y\| \leq \|x\| + \|y\|$
- **Die Komplexen Zahlen** $\mathbb{C} \sim (\mathbb{R}^2, \oplus, \otimes)$
  - multiplication: $\otimes$: $(a, b) \otimes (c, d) := (ac - bd, ad + bc)$
  - addition $(a, b) \oplus (c, d) = (a \oplus c, b \oplus d)$
  - $(1,0) \in \mathbb{R}^2$ ist das neutrale Element von $\otimes$
  - $a + bi = z = re^{i\theta} = r(\cos\theta + i\sin\theta)$
  - $z = a + bi \to \bar{z} = a - bi$
  - $z \cdot \bar{z} = |z|^2$
  - $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$
  - $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$
  - $\sqrt{-n} = i \cdot \sqrt{n}$

2.1 Additional Wisdom

- Injektiv/surjektiv/bijektiv: $f: A \mapsto B,\ r := |\text{Range (f)}| = |A|, i := |\text{Image (f)}| = |B|$. Falls $r \geq i$ kann eine surjektive Abbildung existieren, falls $r \leq i$ kann eine injektive Abbildung existieren, folglich kann eine bijektive Abbildung existieren, falls $r = i$.
- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, für quadratische Gleichungen mit reellen Koeffizienten sind die Lösungen immer entweder beide reell oder es sind zwei zueinander komplex konjugierte Zahlen.

Kapitel 3 Sequences & Series | Folgen & Reihen

- Geometrische Reihe: $\sum_{n=0}^N q^n = \frac{1 + q^{N+1}}{1 - q} \xrightarrow{q<1; N\to\infty} \left(\frac{1}{1-q}\right)$
- Harmonische Reihe: $\sum_{n=1}^\infty \left(\frac{1}{n}\right)^\alpha$
  - konvergiert für $\alpha > 1$
  - divergiert für $\alpha = 1$

3.1 Sequences $(a_n) \subset \mathbb{R}$

- Sei $(a_n)_{n\geq 1}$ eine Folge
- (a_n) heisst **beschränkt**, falls es $c \in \mathbb{R}$ gibt, so dass $|a_n| \leq c\ \forall n \in \mathbb{N}$
- $a_n \underset{n\to\infty}{\longrightarrow} a: \forall\varepsilon > 0, \exists n(\varepsilon) > 0, \text{so that } |a_n - a| < \varepsilon, n > n(\varepsilon)$
- **Important properties**
  - If (a_n) converges, then its limit is unique; can only be used as a negative test (!)
    - $(a_n) = (-1)^n \begin{cases} 1, n \text{ even} \\ 0, n \text{ odd} \end{cases} \to \text{divergent}$
    - (a_n) converges to limit → limit is unique
    - Limit is not unique → (a_n) diverges
  - (a_n) converges → (a_n) bounded; (a_n) is unbounded → (a_n) is divergent; $(a_n) = (n)$ is divergent; (a_n) bounded $\nRightarrow$ (a_n) convergent (see $a_n = (-1)^n$
    - $0 < q < 1, a_n := q^n, \lim a_n = 0$
    - $a_n = \sqrt[n]{n}, \lim a_n = 1$
    - $\lim n^p q^n = 0, p \in \mathbb{N}, 0 < q < 1, t = \frac{1}{q}$, Exponential functions grow faster than any polynomial
- **Convergence criteria** for $(a_n), (b_n)$ convergent
  - $\left.\begin{matrix} \lim a_n = a \\ \lim b_n = b \end{matrix}\right\} \Rightarrow \lim a_n b_n = ab$
  - $\left.\begin{matrix} b_n \neq 0 \\ b \neq 0 \end{matrix}\right\} \Rightarrow \lim \frac{a_n}{b_n} = \frac{a}{b}$
  - $\lim(a_n \pm b_n) = a \pm b$
  - $a_n \leq b_n \forall n \Rightarrow a \leq b$
- **Bionomischer Lehrsatz** $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k$
- Theorem: **Monotone Convergence**

  a_n bounded and monotone increasing $(a_n \leq a_{n+1},\ \forall n \geq 1) \Rightarrow a_n$ convergent and $\lim a_n = \sup\{a_n | n \in \mathbb{N}\}$

  b_n bounded and monotone decreasing $(b_{n+1} \leq b_n,\ \forall n \geq 1) \Rightarrow b_n$ convergent and $\lim b_n = \inf\{b_n | n = 1\}$
  - $a_n = \left(1 + \frac{1}{n}\right)^n \Rightarrow e := \lim \left(1 + \frac{1}{n}\right)^n$
- Monotonkonvergensatz is handy for proving convergence of sequences defined by recursion.
- **Definition** $a \in \mathbb{R}$ ist ein Häufungspunkt von $(a_n)_{n\geq 1}$ falls es eine gegen a konvergeierende Teilfolge $(a_{l(n)})$ gibt
- Sei (a_n) eine beschränkte Folge: **Def** $\liminf a_n := \lim_{k\to\infty} \inf\{a_n : n \geq k\}, \limsup a_n := \lim_{k\to\infty} \sup\{a_n : n \geq k\}$. Dann sind $\limsup a_n, \liminf a_n$ Häfungspunkte von (a_n).
- Theorem: **Bolzano-Weierstrass**: Every bounded sequence has a convergent subsequence
  - There are only finitely many $n \in \mathbb{N}$ with $a_n \notin (a_- - \varepsilon, a_+ + \varepsilon)$
  - a_+ and a_- is the biggest/smallest Häufungspunkt
  - $\liminf a_n, \limsup a_n$
  - $(-1)^n \text{has 2 subsequnces} \begin{cases} (-1)^{2n} = 1 \\ (-1)^{2n+1} = -1 \end{cases}$
- The following are equivalent (TFAE) for (a_n)bounded, $a_- := \liminf a_n, a_+ := \limsup a_n$
  - (a_n) converges to a
  - Every subsequence converges to a
  - $a_- = a_+$
- **Cauchy-Criteria**: $a_n \subset \mathbb{R}$: a_n conv.⇔ $a_n$is Cauchy
- **Cauchy**: $\forall\varepsilon > 0, \exists n(\varepsilon)$ so that $|a_n - a_m| < \varepsilon, n, m > n(\varepsilon)$
  - $a_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, a_n is divergent, we showed it is not Cauchy

3.2 Series

- (a_n) sequence $\Rightarrow s_n := a_1 + a_2 + \cdots + a_n$
- $\sum_{k=1}^{\infty} a_k$ convergent ⇔ $\lim s_n$ exists, $\lim \sum_{k=1}^{n} a_n$ exists
- $\lim_{n\to\infty} s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k$

- **Convgerence criteria for series**
  - $\sum a_k \text{conv.} \Leftrightarrow |\sum_{k=n}^{m} a_k| < \varepsilon; n, m > n(\varepsilon)$
  - $\sum a_k \text{conv.} \Rightarrow \lim a_n = 0; \lim a_n \neq 0 \Rightarrow \sum a_k$ divergent

    Warning: $\lim a_n = 0 \nRightarrow \sum a_k$ conv. e. g. $a_n = \frac{1}{n}, \lim a_n = 0, \sum \frac{1}{n}$ div; e. g. $a_n = q^n, \lim a_n = 0, \sum q^n$ conv.
  - Majoranten (Minoranten) Kriterium (Comparison tests)

    Let $\sum a_k, < \sum b_k$ series so that

$$\left.\begin{matrix}\exists k_0 \text{ so that } |a_k| \leq b_k \; \forall k \geq k_0 \\ \sum b_k \text{ conv.}\end{matrix}\right\} \Rightarrow \sum a_k \text{conv.}$$

$$\left.\begin{matrix}\exists k_0 \text{ so that } a_k \geq b_k > 0 \; \forall k > k_0 \\ \sum b_k \text{ div.}\end{matrix}\right\} \Rightarrow \sum a_k \text{ div.}$$

    → $\sum \frac{1}{k!}, \sum \frac{1}{(k+1)^2}$ conv. $< \sum \frac{1}{2^k} \leq \sum \frac{1}{k(k+1)}$
  - Ratio test (Quotientenkriterium)
    1. If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_k$ conv.
    2. If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\sum a_k$ div.
    3. When these limits are 1 → no information

$$\rightarrow \quad \left.\begin{matrix}\sum \frac{1}{n} \text{ div.} \\ \sum \frac{1}{n^2} \text{ conv.}\end{matrix}\right\} \lim \frac{a_{n+1}}{a_n} = 1$$

$$\rightarrow \quad \sum_{k=0}^{\infty} \frac{z^k}{k!} \text{ conv. for every } z; = Exp(x)$$

  - Root test
    1. $\limsup \sqrt[n]{a_n} < 1 \Rightarrow$ conv.
    2. $\limsup \sqrt[n]{a_n} > 1 \Rightarrow$ div.
    3. When it is 1 → no information

$$\rightarrow \quad \xi(\alpha) := \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \begin{cases} \text{conv}, \alpha > 1 \\ \text{div}, \alpha \leq 1 \end{cases}; \int_1^{\infty} \frac{1}{x^\alpha dx} = \begin{cases} < \infty \text{ if } \alpha > 1 \\ \infty \text{ if } \alpha \leq 1 \end{cases}$$

- **Convergence radius**: $r = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$
- **Absolute convergence**: $\sum |a_k|$ converges:
  - → $\sum a_k$ abs conv $\Rightarrow \sum a_k$ conv.
  - → $\sum a_k$ conv. $\nRightarrow \sum a_k$ abs conv.
  - → $\sum \frac{(-1)^n}{n}$ conv. but $\sum \frac{1}{n}$ is div.
- Importance of abs. conv. Is that we can reorder the terms in the sum the way we want; formally: let $\sum_{k=1}^{\infty} a_k$ be absolut convergent, $\varphi: \mathbb{N} \to \mathbb{N}$ a bijection. Then $\sum_{k=1}^{\infty} a_{\varphi(k)}$ converges absolutely and has the same sum value.

$$\rightarrow \quad \sum \frac{z^k}{k!} \xRightarrow[\text{abs.conv.}]{} \text{Exp}(x+y) = \text{Exp}(x) Exp(y)$$

3.3 Additional Wisdom

- Bei einer Funktion bei der das Vorzeichen alterniert und der der Betrag gegen unendlich geht (à la $\left(-\frac{3}{2}\right)^n$), existiert der Limes nicht (für $(-1)^n$ gilt ähnliches).

Kapitel 4 Continuity, Limits of Functions

- f has a limit of a at $x = x_0$
- $\lim_{x\to x_0} f(x) = a$ if for every (x_n) with $\lim x_n = x_0, \lim f(x_n) = a$

  f is continuous at x_0 if
  - $f(x_0)$ is defined
  - $\lim_{x\to x_0} f(x)$ exists
  - $\lim_{x\to x_0} f(x) = f(x_0)$
- $f(\lim x_n) = \lim f(x_n)$
- Continuity behaves with respect to operations on functions:

$$f, g \text{ cont at } x_0 \Rightarrow f+g, fg \text{ cont at } x_0$$
$$if\ g(x_0) \neq 0 \Rightarrow {}^{f}/_{g} \text{ is cont}$$

- f cont at $x_0 \equiv \forall \varepsilon > 0, \exists \delta_{\varepsilon,x_0} > 0$ so that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$
- f is cont on Ω
$$\forall x_0 \in \Omega, \forall \varepsilon > 0, \exists \delta_{\varepsilon,x_0} > 0 \text{ so that } \forall x \in \Omega\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$
- **Uniform continuity**:
$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ so that } \forall x, x_0 \in \Omega |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$
- If f is cont on a compact set then it is uniform cont on K → $[a, b]$
- **Important properties of continuous functions**
  - $f: [a, b] \to \mathbb{R}$, cont $\Rightarrow f([a, b])$ is bounded and $\exists c_+, c_- \in [a, b]$ so that $f(c_+) = \sup f$, $f(c_-) = \inf f$
  - **Zwischenwertsatz**: if $a < b, f: [a, b] \to \mathbb{R}$ cont with $f(a) < f(b)$ $\big(or\ f(a) > f(b)\big) \Rightarrow$ for every $y \in [f(a), f(b)], \exists c \in [a, b]$ such that $f(c) = y$; Korollar: jedes Polynom mit einem ungeraden Grad besitzt mindestens eine reelle Nullstelle.
  - $f: [a, b] \to \mathbb{R}$ cont, strict monotone $\Rightarrow$ Bild $f = [c, d] = [f(a), f(b)]$ and $f: [a, b] \to [c, d]$ is bijective and $f^{-1}: [c, d] \to [a, b]$ is continous
    Log is continous, inverse of monotone, cont function e^x

4.1 Pointwise and uniform convergence of sequences of functions

- Let (f_n) be a sequence of functions, f another function
- Pointwise convergence: $f_n \xrightarrow{p.w.} f$ if $\forall x \in \Omega \lim_{n\to\infty} f_{n(x)} = f(x)$; i.e. $\forall x \in \Omega, \forall \varepsilon > 0, \exists k_{\varepsilon,x}$ sd $\forall k > k_{\varepsilon,x} |f_k(x) - f(x)| < \varepsilon$
  In pointwise conv one can have a sequence f_n of continuous functions with limit f discontinuous; "cure" → uniform convergence
    - $f_n = x^n: [0,1] \to \mathbb{R}, \lim_{n\to\infty} x^n = \begin{cases} 0 \text{ if } 0 \le x < 1 \\ 1 \text{ if } x = 1 \end{cases}$
- Uniform convergence of sequences of functions: $f_k \xrightarrow{\text{uniform}} f$ if $\sup_{x\in\Omega} \left| f_{k(x)} - f(x) \right| \xrightarrow[k\to\infty]{} 0$, i.e. $\forall \varepsilon > 0\ \exists k_\varepsilon$ sd $\forall k > k_\varepsilon, \forall x \in \Omega\ |f_k(x) - f(x)| < \epsilon$
- $f_k \xrightarrow{\text{uniform}} f \Rightarrow f_k \xrightarrow{\text{pointwise}} f$, but NOT vice versa
- Theorem: If $f_k \to f$ uniformly, f_k are continuous then f is continuous

Kapitel 5 Differentialrechnung

- f is differentiable in x_0 if $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists
- $f'(x) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$
- **Theorem 5.5**: f diff. in $x_0 \Rightarrow f$ cont. in x_0; f cont. in $x_0 \not\Rightarrow f$ diff. in x_0
  - e.g. $f(x) = |x|$
- **Rules**, $f, g\ diff.in\ x_0$
  - $f + g, fg$, if $g(x_0) \neq 0 \frac{f}{g}$ diff
  - $(f + g)' = f' + g'$
  - $(fg)' = f'g + fg'$
  - $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
- **Chain rule**: f diff in x_0, g diff in $f(x_0) \Rightarrow g \circ f$ diff in x_0 and $(g \circ f)'(x_0) = g'\big(f(x_0)\big)f'(x_0)$
- Theorem 5.12: $f: [a, b] \to$ cont in $\mathbb{R}$ and diff on (a, b)
  Let $z_+ \in [a, b]$ with $f(z_+) = \underset{x\in[a,b]}{\text{Max}} f(x) \Rightarrow f'(z_+) = 0$
- Theorem 5.14 (**Mittelwertsatz**): Let $f: [a, b]$ be cont $\Rightarrow$ diff on $(a, b), a \neq b$
  $\Rightarrow \exists x_0 \in (a, b)$ with $f'(x_0) = \frac{f(b)-f(a)}{b-a}$
- Corollary: f as in Th 5.14 (e.g. $f' = \lambda f \Rightarrow f(x) = ce^{\lambda x}$

- $f'(x) = 0\ \forall x \in (a,b) \Rightarrow f(x) = \text{const}$
- $f'(x) \geq 0\ \forall x \in (a,b) \Rightarrow f(x)$ mon. inc.
  $f'(x) > 0\ \forall x \in (a,b) \Rightarrow f(x) =$ strictly mon. inc.
- $f'(x) \leq 0\ \forall x \in (a,b) \Rightarrow f(x)$ mon. dec.
  $f'(x) < 0\ \forall x \in (a,b) \Rightarrow f(x) =$ strictly mon. dec.

– Corollary: **L'Hospital**: $f, g: [a,b]$ cont, diff in $(a,b)$with $g'(x) \neq 0\ \forall x \in (a,b)$
  Assume: (1) $f(a) = 0 = g(a)$ and (2) $\lim_{x \downarrow a} \frac{f'(x)}{g'(x)} = A$ Then $g(x) \neq 0\ \forall x > a$ and $\lim_{x \downarrow a} \frac{f(x)}{g(x)} = A$
– **Umkehrsatz**: $f: (a,b) \to \mathbb{R}$ diff with $f'(x) > 0\ \forall x \in (a,b)\big(or\ f'(x) < 0\ \forall x \in (a,b)\big)$, let $c = \inf f(x)$, $d = \sup f(x)$. Then $f: (a,b) \to (c,d)$is bjective and the inverse function $f^{-1}: (c,d) \to (a,b)$ is diff with $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}\ \forall y \in (c,d)$, Examples
  - $(\log x)' = \frac{1}{x}$
  - $x^\alpha := e^{\alpha \log x}, (x^\alpha)' = \alpha x^{\alpha - 1}$
  - $(\arcsin x)' = 1/\sqrt{1 - x^2}, (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, (\arctan x)' = \frac{1}{1+x^2}$

– Functions of the class C^m, $C^m(\Omega)$: $\{f: \Omega \to \mathbb{R} | f^0, f', f'', \ldots, f^m$ exist and are cont$\}$
– Theorem: $f_n \subset C'(\Omega), f_n, f_n'$ cont, $f_n \rightrightarrows f$ and $f_n' \rightrightarrows g$ then $f \in C'(\Omega)$ and $f' = g$
– Cor 5.32: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with convergence radius ρ then $f(x) \subset C^\infty(-\rho, \rho)$ and die Ableitungen von f erhält man durch gleidweises differenzieren.

$$f^k(x) = \sum_{n=k}^{\infty} a_n \frac{n!}{(n-k)!} x^{n-k}$$

– **Taylor**: Let $f \in C^{n-1}(\Omega), in[a,b] \subset \Omega\ f$ is $n -$ times differentiable, $x_0, x \in [a,b]$. Then $\exists c \in (x_0, x)$ with $f(x) = f(x_0) + f'(x_0)\frac{x - x_0}{1!} + \cdots + f^{m-1}(x_0)\frac{(x-x_0)^{m-1}}{(m-1)!} + f^m(c)\frac{(x-x_0)^m}{m!}$

$$f(x) = Tm(x, x_0) + \mathrm{Rm}\ f(x; x_0)$$
$$\mathrm{Tm}(f, x_0) = f(x_0) + \cdots + f^m(x_0)\frac{x - x_0}{m!}$$
$$\mathrm{Rm}\ f = f(x) - \mathrm{Tm}(x, x_0) = \frac{f^m(c) - f^m(x_0)}{m!}(x - x_0)^m$$
$$|\mathrm{Rm}\, f| \leq \left(\sup_{x_0 < \xi < x} |f^{n+1}(\xi)|\right)\frac{(x - x_0)^{n+1}}{m!}$$

5.1 Local extrema

– Theorem 5.39: $\Omega \subseteq \mathbb{R} f: \Omega \to \mathbb{R}; f \in C^\infty(\Omega) x_0 \in \Omega$. Then we'll have one of the following cases:
  - $f^j(x_0) = 0\ \forall j > 1$
  - $f'(x_0) = 0 = f''(x_0) = \cdots f^{m-1}(x_0) = 0\ and\ f^m(x_0) \neq 0$
    2.1 m odd $\Rightarrow x_0$ is no extrema
    2.2 m even and $f^m(x_0) > 0 \Rightarrow x$ is a strict local minimum
    2.3 m even and $f^m(x_0) < 0 \Rightarrow x$ is a strict local maximum

– Convex functions: $f: (a,b) \to \mathbb{R}$ is convex if $\forall x_0 \leq x \quad t \in [0,1]$ such that $f(tx_0 + (1-t)x_1) \leq tf(x_0) + (1-t)f(x_1)$. The graph of f lies below every possible line of two of its points
– Theorem: $f: (a,b) \to \mathbb{R}$ of class C^2 with $f''(x) \geq 0\ \forall x \in (a,b) \Rightarrow f$ is convex
– Jensen's inequality: $f: (a,b) \to \mathbb{R}$ convex, $\forall x_1, \ldots x_n \in (a,b)$ and $t_1, \ldots, t_n \in [0,1]$ with $\sum t_i = 1$ the following is true

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i)$$

5.2 Additional Wisdom

– Es gilt stets, dass das Taylorpolynom n-ter Ordnung eines Polynoms von Grad kleiner oder gleich n gleich dem Polynom selbst ist, da die $n+1$-te Ableitung gleich Null ist und somit der Restterm r_{n+1} Null ist.