Lecture Summary

Note This document was written in the following way (which might explain its structure): First, I copied Prof. Imamoglu's non-example notes for the review. Then I merged the in-class examples from the review and our assistant's (Tim) review. The language is inconsistent.

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1 Integration

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function, $P = \{a = x_o < x_{1x} < \cdots < x_n = b\}$ a partition of the interval $[a, b]$ and $\xi_k \in [x_k, x_{k+1}]$ points in each subinterval. Then the sum $S(f, P, \xi) = \sum_{k=0}^{n-1} f(\xi_k)(x_{k+1} - x_k)$ is called the **Riemann sum** attached to f and to P. For $I_k = [x_k, x_{k+1}]$, $U(f, P) = \sum_{k=0}^{n-1} \left(\inf_{I_k} f\right)(x_{k+1} - x_k)$ and $O(f, P) = \sum_{k=0}^{n-1} \left(\sup_{I_k} f\right)(x_{k+1} - x_k)$ are called the lower and upper Riemann sums. Similarly $\underline{\int_a^b} f\, dx = \sup\{U(f, p), p \in P(I)\}$ and $\overline{\int_a^b} f\, dx = \inf\{O(f, p), p \in P(I)\}$ are called lower and upper integrals. f is called Riemann integrable if $\underline{\int_a^b} f\, dx = \overline{\int_a^b} f\, dx$.

1.1 Facts

- Each continuous function is Riemann integrable
- Each monotonic function is Riemann integrable

1.2 Properties

- Let f, g Riemann integrable on I, $\alpha, \beta \in \mathbb{R}$. Then
  1. $\int_a^b (\alpha f + \beta g) dx = \alpha \int_a^b f\, dx + \beta \int_a^b g\, dx$
  2. If $f(x) \leq g(x)\ \forall x \in [a, b]$ then $\int_a^b f\, dx \leq \int_a^b g\, dx$
  3. $\left|\int_a^b f\, dx\right| \leq \int_a^b |f(x)| dx$
  4. $\left(\inf_I f\right)(b - a) \leq \int_a^b f(x)\, dx \leq \left(\sup_I f\right)(b - a)$
  5. $\int_a^b f\, dx = -\int_b^a f(x)\, dx$
  6. $\int_a^b f\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx\ \forall a, b, c \in \mathbb{R}$
  7. $f(x) \geq 0\ \forall x \in D \Longrightarrow \int_D f\, d\mu \geq 0$

1.3 Mittelwertsatz der Integralrechnung

$f: [a, b] \to \mathbb{R}$ continonous. Then $\exists \xi \in [a, b]$ such that $\int_a^b f(x)\, dx = f(\xi)(b - a)$.

1.4 Fundamental theorem of Calculus

1. Let $f: [a, b] \to \mathbb{R}$ continuous. Define $F(x) := \int_a^x f(t)\, dt\ \ \forall x \in [a, b]$. Then F is differentiable and $F' = f$. F is called a **primitive (Stammfunktion)** of f
2. If G is an anohte rpmimirve of f then $G = F + c$ for some constant c
3. Let F be any pmirmitve of f, then $\int_a^b f(x)\, dx = F(b) - F(a)$

1.5 How do we calculate integrals?

Never forget the constant!

1. Partial integration: follows product rule for differentiation
   $\int f(x) g'(x)\, dx = f(x) g(x) - \int f'(x) g(x)\, dx$
   $\int_a^b f(x) g'(x)\, dx = f(x) g(x)|_a^b - \int_a^b f'(x) g(x)\, dx$
2. Substitution: follows chain rule for differentiation
   $\int f(x)\, dx = \int f(\varphi(y)) \varphi'(y)\, dy$
   $\int_{\varphi(a)}^{\varphi(b)} f(x)\, dx = \int_a^b f(\varphi(y)) \varphi'(y)\, dy$
3. Partial fractions: to integration rational functions of the form $\frac{P(x)}{Q(x)}$, P, Q are polynomials
   $\frac{P(x)}{Q(x)} = \frac{P(x)}{(x^2+1)(x-1)^2(x+2)}$
   $\to$ Ansatz: $\frac{P(x)}{Q(x)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{x+2}$
   Beachte Vielfachheiten, $\mathbb{C}$ – Nullstellen

– Basic types of integration of rational functions
  › Polynomial: $\int \sum a_n x^n\, dx = \sum a_n \frac{x^{n+1}}{n+1} + c$

- Inverse powers: $\int \frac{dx}{(x-x_0)^r} = \begin{cases} \log|x-x_0|, & \text{für} = 1 \\ \frac{1}{1-r}\frac{1}{(x-x_0)^{r-1}}, & \text{für} \geq 2 \end{cases}$

1.6 Improper Integrals

The improper integral of an integrable function f on (a,b) which is integrable on any subinterval $[a',b']$. We define the improper integral $\int_a^b f(x)\,dx := \lim_{a'\searrow a}\lim_{b'\nearrow b}\int_{a'}^{b'} f(x)\,dx$.

“Integralkriterium”

- $f(x)$ is defined on $[a,\infty[$
- $f(x) \geq 0\ \forall x \in [a,\infty[$
- $f(x)$ monoton fallend $\Leftrightarrow f'(x) \leq 0$
- $\Longrightarrow \sum_{n=a}^{\infty} f(n)$ konvergiert $\Leftrightarrow\ \int_a^\infty f(x)\,dx$ konvergiert

Facts

1. $\forall s \in \mathbb{R}, a > 0,\ \int_a^\infty \frac{dx}{x^s} = \begin{cases} \frac{a^{1-s}}{s-1}, & s > 1 \\ \infty, & s \leq 1 \end{cases}$
2. If f is on $[a,\infty)$ continuous and $\exists c$ and $s > 1$ so that $|f(x)| \leq c/x^s\ \forall x \geq a$, then $\int_a^\infty f(x)\,dx$ converges
3. If f is in $[a,\infty)$ continuous and $\exists c > 0$ such that $f(x) \geq c/x,\ \forall x \geq a$, then $\int_a^\infty f(x)\,dx$ diverges to ∞.

1.7 Examples

1.7.1 Examples BP 2013

- $\int_1^2 \frac{\sqrt{1+\ln x}}{x} dx$

$$u = 1 + \ln x$$
$$du = \frac{dx}{x}$$
$$\int_1^2 \frac{\sqrt{1+\ln x}}{x} dx = \int_1^{1+\ln 2} u^{1/2} du = \frac{u^{3/2}}{3/2}\ from\ 1\ to 1 + \ln 2$$

- $\int \cos x \cosh x\, dx$

$$u = \cos x \qquad v' = \cosh x$$
$$u' = -\sin x \qquad v = \sinh x$$
$$I = \int \overbrace{(\cos x)}^{u}\overbrace{\cosh x}^{v'} dx = uv - \int vu'$$
$$= (\cos x)(\cosh x) + \int \overbrace{(sin\ x)}^{u}\overbrace{sinh\ x}^{v'} dx$$
$$= \cos x \cosh x + \left[\sin x \cosh x - \int \cos x \cosh x\, dx\right]$$
$$I = \frac{1}{2}[\cos x \sinh x + \sin x \cosh x] + C$$

- $\int \frac{x^2-x+2}{x^3-x^2+x-1}$

$$x^3 - x^2 + x - 1 = x^2(x-1) + (x-1) = (x-1)(x^2+1)$$
$$\int \frac{x^2-x+2}{x^3-x^2+x-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$
$$\Rightarrow A(x^2+1) + (Bx+C)(x-1) = x^2 - x + 2$$
$$\Rightarrow A = 1, C = -1, B = 0;\ \int \frac{x^2-x+2}{x^3-x^2+x-1} = \int \frac{1}{x-1} - \frac{1}{x^2+1} dx$$
$$= \ln x - 1 - \tan^{-1} x + C$$

1.7.2 Example Spring 2010

$$\text{Untersuche, ob das untere Integral } \int_1^\infty \frac{1}{x^2+x} dx \text{ konvergiert.}$$
$$-x^2 + x \geq x^2 \rightarrow \int_1^\infty \frac{1}{x^2+x} \leq \int_1^\infty \frac{1}{x^2}\ dx \rightarrow \text{converges}$$

or

$$\lim_{b\to\infty}\int_1^b \frac{1}{x(x+1)}dx = \lim \int_1^b \frac{1}{x} - \frac{1}{x+1}dx = \lim[\ln|x| - \ln|x+1|]^b$$
$$= \lim_{b\to\infty} \ln\left|\frac{x}{x+1}\right|_1^b = \lim_{b\to\infty} \ln\frac{b}{b+1} - \ln\frac{1}{2} = \ln 2$$

1.8 Additional Wisdom

- From http://en.wikipedia.org/wiki/Differentiation_under_the_integral_sign:
  - $\frac{\partial}{\partial b}\left(\int_a^b f(x)\,dx\right) = f(b), \frac{\partial}{\partial a}\left(\int_a^b f(x)\,dx\right) = -f(a)$
  - $\varphi(\alpha) := \int_a^b f(x,\alpha)\,dx, \frac{d\varphi}{d\alpha} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx + f(b,\alpha)\frac{\partial b}{\partial\alpha} - f(a,\alpha)\frac{\partial a}{\partial\alpha}$ (Leibniz)
- $n! \approx \sqrt{(2\pi n)}\left(\frac{n}{e}\right)^n$ (Stirling)
- Solids of revolution
  - when integrating *parallel* to the axis of revolution: $V = \pi\int_a^b f^2(x)\,dx$
  - when integrating *perpendicular* to the axis of revolution: $V = 2\pi\int_a^b x|f(x)|\,dx$
- Don't forget $+C$ when integrating

2 Differential Equations

2.1 Linear differential equations with constant coefficients

To solve a linear differential equations of the form $Ly' = b(x)$ where $L := \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \dots + a_1\frac{d}{dx} + a_0$, $b(x)$ a function, $a_i \in \mathbb{R}$.

1. Find a homogenous solutiony_H. Namely a solution of $Ly = 0$.
2. Find a special solution y_S of $Ly = b(x)$ using the method of "Ansatz vom Typ der rechten Seite".
3. The general solution is given by $y = y_H + y_S$

2.1.1 Finding the homogeneous solution y_H of $Ly = 0$

1. Find the characteristic polynomial of L. Namely $P_L(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$
2. Fact: if $\lambda_1, \dots \lambda_r \in \mathbb{C}$ are the pairwise distinct roots of $p(\lambda) = 0$ with associated multiplicities $m_1, \dots, m_R$, then the functions $x \to x^k e^{\lambda_j x}$, $1 \le j \le r, 0 \le k \le m_j$ form a system of fundamental solutions of the homogenous equation $Ly = 0$.
   Note: if L has real coffeicients, every pair of cimplex conjugate, non-ral roots $\lambda_j = \mu_\lambda \pm i\nu_j$ of multiplicity m_j give a fundamental solution $x^k e^{(\mu_j \pm i\nu_j)x} = x^k e^{\mu_j}\left(\cos\nu_j x \pm i\sin\nu_j x\right)$ for $0 \le k < m_j$. So one can as a basis take $x^k e^{\mu_j x}\cos\nu_j x$ and $x^k e^{\mu_j x}\sin\nu_j x$ instead of $x^k e^{(\mu_j + i\nu_j)x}$ and $x^k e^{(\mu_j - i\nu_j)x}$. Then the general homogenous solutions is of the form $y_H(x) = \sum_{j=1}^r \sum_{k=0}^{m_j} c_{jk} x^k e^{\lambda_j x}$ with constants c_{jk}.

2.1.2 How to find the special solution of $Ly = b(x)$ using the method of "Ansatz"

Facts

1. Let $\lambda \in \mathbb{C}$. If λ is not a solution of $p_L(\lambda) = ??$, then the inhomongoues DGL $Ly = e^{\lambda x}$ has particular solution $y = \frac{1}{p_L(\lambda)}e^{\lambda x}$
2. Let $\lambda \in \mathbb{C}$, m its multiplicity as a solution of $p_{L(\lambda)} = 0$ (m can be zero which means λ is not a solution of $p_L(\lambda) = 0$). Let $Q(x)$ a polynomial of degree k. Then a particular solution of $Ly(x) = Q(x)e^{\lambda x}$ is of the form $y(x) = R(x)e^{\lambda x}$ for a polynomial $R(x)$ of degree $k + m$
3. If L has real coefficients. Let $\mu, \nu \in \mathbb{R}, m$ the multiplicity of $\mu \pm i\nu$ as a solution of $p_L(\lambda) = 0$ ($m = 0$ means $\mu \pm i\nu$ is a root of p_L). Let $Q(x), R(x)$ be a polynomial of degree$\le k$. The particular solution of the inhomogeneous DGL $Ly = Q(x)e^{\mu x}\cos\nu x + R(x)e^{\mu x}\sin x$ is of the form $y(x) = s(x)e^{\mu x}\cos\nu x + T(x)e^{\mu x}\sin x$ for polynomials S, T of degree $\le k + m$

2.2 Boundary or initial value problems

When we are given a DGL $Ly = b(x)$ together with either boundary values $\begin{matrix} y(a_1) = A_1 \\ y(a_2) = A_2 \\ \dots \\ y(a_n) = A_n \end{matrix}$ or initial values $\begin{matrix} y(0) = A_1 \\ y'(0) = A_2 \\ \dots \\ y^{n-1}(0) = A_n \end{matrix}$, we first find the general solution $y = y_H + y_S$. Then we determine the constants $c_1, \dots, c_n$in the homogenous solution using the given boundary/initial values.

2.3 Solving DGL by separation of variables

Facts

- If $f: \Omega \to \mathbb{R}$ is differentiable in $x_0 \in \mathbb{R}$, then the partial derivatives exists and the differential $df(x_0)$ has the matrix representation $\left(\frac{\partial f}{\partial x'}(x_0) \quad \dots \quad \frac{\partial f}{\partial x^n}(x_0)\right) = \nabla f$ the gradient of f.
- f diff in $x_0 \Longrightarrow f$ is continous in x_0
- If all partial derivatives of f exists *and* continuous, then f is differentiable.

Using the last two facts and the definition of differentiability, one can study if a given is differentiable of not.

Recipe

- Sei die DG in der Form $\frac{df(x)}{dx} = g(x)h\big(f(x)\big)$
- Sei nun $y = f(x)$, dann $\frac{dy}{dx} = g(x)h(x)$
- Falls $h(y) \neq 0$, dann $\frac{dy}{h(y)} = g(x)dx$
- Alternative Notation: $\frac{1}{h(y)}\frac{dy}{dx} = g(x)$
- werden nun beide Seiten nach x integriert, dann $\int \frac{1}{h(y)}\frac{dy}{dx}dx = \int g(x)\,dx \Leftrightarrow \int \frac{1}{h(y)}dy = \int g(x)\,dx$
- $\int \frac{y'}{y}\,dy = \ln|y|$

2.4 Ansätze

$a, b, c, d \in \mathbb{R}, \mu, \nu \in \mathbb{R}, n \in \mathbb{N}, X_n = \text{Polynomial of degree } x$

Störfunktion $q(x)$	Ansatz für $y_p(x)$
$ae^{\mu x}$	$be^{\mu x}$
$a \sin \nu x$ $b \cos \nu x$	$c \sin \nu x + d \cos \nu x$
$ae^{\mu x} \sin \nu x$ $be^{\mu x} \cos \nu x$	$e^{\mu x}(c \sin \nu x + d \cos \nu x)$
$P_n(x)e^{\mu x}$	$R_n(x)e^{\mu x}$
$P_n(x)e^{\mu x} \sin \nu x$ $Q_n(x)e^{\mu x} \cos \nu x$	$e^{\mu x}(R_n(x) \sin \nu x + S_n(x) \cos \nu x)$

Bem 1 Liegt eine Linearkombination der Störfunktionen vor, so hat man auch als Ansatz eine entsprechende Linear-kombination zu wählen.

Bem 2 Falls $\tilde{\lambda} = \mu + i\nu$ eine m–fache Nullstelle des charakteristischen Polynoms von (H) ist, so muss man den Ansatz für $y_p(x)$ mit dem Faktor x^m multiplizieren.

2.5 Examples

2.5.1 Example Spring 2011

a) Bestimme alle Lösungen $y = y(x)$ der DGL $y^{(4)} - y = 0$ welche für $|x| \to \infty$ beschränkt bleiben.

$$Characteristic\ polynomial: x^4 - 1 = 0 \Leftrightarrow (x^2 - 1)(x^2 + 1) = 0$$

$$\Rightarrow \begin{matrix} \lambda = \pm 1 & \to e^x, e^{-x} \\ \lambda = \pm i & \to \cos x, \sin x \end{matrix}$$

$$y_H(x) = c_1 e^x + c_2^{-x} + c_3 \cos x + c_4 \sin x$$

The solutions that remain bounded as $|x| \to \infty$ are of the form $c_3 \cos x + c_4 \sin x$

b) Bestimme eine Lösung $y = y(x)$ der DGL $y^{(4)} - y = e^{-x} + x$

$y^4 - y = e^{-x} \to y_{p_1}$

$y^4 - y = x \to y_{p_2}$

Superposition: $y_p = y_{p_1} + y_{p_2}$

$y = c_1 e^x + c_2 e^{-x} + \cdots + \widetilde{b(x)}$

$y_{p_1} = Cxe^{-x}$ and $y_{p_2} = Dx + E$

Try $y_p = Cxe^{-x} + Dx + E$, put this in $y^4 - y = e^x + x$

$y_p^{(4)}(x) = C[-4e^{-x} + xe^{-x}]$

$y_p^{(4)}(x) - y_p(x) = C[-4e^{-x} + xe^{-x}] - [Cxe^{-x} + Dx + E] = e^{-x} + x$

$\Rightarrow c = \frac{1}{4}, D = -1$

2.5.2 Example Summer 2013

a) Für welche Werte des Paramaters $a \in \mathbb{R}$ strebt die allgemeine Lösung der DGL $y'' + 2y' + ay = 0$ unabhängig von den Anfangsbedingungen gegen 0 für $x \to \infty$?

$\lambda^2 + 2\lambda + a = 0 \Rightarrow \lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 4a}}{2} = -1 \pm \sqrt{1 - a}$

For $1 - a < 0$: there are 2 complex conjugate roots. Let $|1 - a| = b^2. Then\ -1 \pm bi$

$\to c_1 e^x \cos bx + c_2 e^{-x} \sin bx \to 0$ as $x \to \infty$

For $1 - a = 0$: (-1) is a double root. The solution $c_1 e^{-x} + c_2 e^{-x} x$

$\to 0$ independeant of the initial conditions.

For $1 - a > 0$: then one of the roots will be positive **if** $\sqrt{1 - a} > 1$. That will lead to $\lambda = -1 + \sqrt{1 - a}$

> 0 which leads to a growing solution. We do not want $1 - a > 1$ or $a < 0$. **If** $\sqrt{1 - a}$

< 1 then $\lambda_{1,2} < 0$

b) Finden Sie eine homogene DGL 2. Ordnunug mit konstanten Koeffizienten, deren allgemeine Lösung $y(x) = e^{-x} + 2xe^{-x}$ ist. Was sind dann die Anfangsbedingungen bei $x = 0$?

$y = e^{-x} + 2xe^{-x}$

We are looking for a 2nd DGL. By looking at the equation, you can see that λ

$= -1$ with multiplicity 2 (i. e. double root of the char. pol.). $\to (\lambda + 1)^2 = \lambda^2 + 2\lambda + 1$

$y'' + 2y' + y = 0 +$ initial values $\Rightarrow c_1 = 1, c_2 = 2$

$\underbrace{y_{GH}}_{general\ homogenous\ solution} = c_1 e^{-x} + c_2 x e^{-x}$

$y(0) = e^{-0} = 1\ and\ y'(0) = -e^{-x} + 2[e^{-x} - xe^{-x}]$ for $x = 0 \to = 1$

2.5.3 Example Spring 2011

Bestimme die Lösung $y = y(x)$ der DGL $y' = e^{x-y}$ mit $y(0) = 0$

$\Rightarrow y' = \frac{e^x}{e^y} \Rightarrow dye^y = e^x dx \Rightarrow \int e^y dy = \int e^x dx \Rightarrow e^y = e^x + c$

$y = \ln e^x + c$

$0 = y(0) = \ln e^0 + c = \ln 1 + c \Rightarrow c = 0$

3 Differentiation in $\mathbb{R}^n$

A function $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable in x_0 if there exists a linear map $A: \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = f(x - x_0) + A(x - x_o) + R(x, x_0)$ where $\lim_{x \to x_0} \frac{R(x,x_0)}{|x - x_0|} = 0$. In this case A is called the differential of f at x_0 and it is dented by $(df)(x_0)$.

Let $(A_1, A_2, \dots, A_n)$ be a matrix representation of the linear map $A: \mathbb{R}^n \to \mathbb{R}$. The f differentiable at x_0 means $f(x) = f(x_0) + A_1(x^1 - x_0^1) + A_2(x^2 - x_0^2) + \cdots + A_n(x^n - x_0^n) + R(x, x_0)$.

A partial derivative is defined as $\frac{\partial f}{\partial a_i}(\vec{a}) \coloneqq \lim_{h \to 0} \frac{f(a_1, .., a_{i-1}, a_i + h, a_{i+1}, \dots, a_n) - f(a_1, \dots, a_i, \dots, a_n)}{h}$

Fact Let $h(s, t)$ be a continuously differentiable function of two variables and $b(t)$ a differentiable function of one variable. Define $u(t) \coloneqq \int_a^{b(t)} h(s, t)\ ds$. Then u is diffenetbale and $u'(t) = h(b(t), t) \cdot b't(t) + \int_a^{b(t)} \frac{\partial h}{\partial t}(s, t)\ d??$

In particular if $u(t)$ is defined as a definite integral of $h(s,t)$ in the variable s, $u(t) := \int_a^b h(s,t)\,ds$, then u is differentiable and one can interchange the order of differentiation and integration. That is $\frac{d}{dt}u(t) = \frac{d}{dt}\int_a^b h(s,t)\,ds = \int_a^b \frac{\partial}{\partial t} h(s,t)\,ds$.

3.1 Differentiation rules

Let $f, g: \Omega \to \mathbb{R}$ differentiable in x_0. Then:

1. $d(f \pm g)(x_0) = df(x_0) + dg(x_0)$
2. $d(fg)(x_0) = g(x_0)df(x_0) + f(x_0)dg(x_0)$
3. If $g(x_0) \neq 0$ then $d(f/g)(x_0) = \frac{g(x_0)fd(x_0) - f(x_0)dg(x_0)}{(g(x_0))^2}$
4. Let $h: \mathbb{R} \to \mathbb{R}$ be differentiable in $g(x_0)$. Then $d(h \circ g)(x_0) = h'(g(x_0)) \cdot dg(x_0)$
5. Let $H: I \subset \mathbb{R} \to \Omega \subset \mathbb{R}^n$ be differentiable in $x_0 \in I$ and $f: \Omega \to \mathbb{R}$ differentiable in $H(t_0)$. Then $\frac{d}{dt}(f \circ H)(t_0) = df(H(t_0)) \cdot H'(t_0)$ where $H(t) = (H_1(t), H_2(t), \dots, H_n(t))$, $H'(t) = (H_1'(t), H_2'(t), \dots, H_n'(t))$
6. Chain rule: $\frac{d}{dt}(f \circ g)(t_0) = df(g(t_0)) \cdot g'(t_0)$; $\frac{d}{dt}\left(f(x(t), y(t))\right) = df(x(t), y(t)) \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot y'(t)$

3.2 Directional derivative

The directional derivative of f is in the direction of a unit vector $e \in \mathbb{R}^n - \{0\}$ is given by $d_e f(x_0) = \nabla f(x_0) \cdot \vec{e}$.

3.3 Higher partial derivatives

One can similarly define higher order partial derivatives for functions $f \in C^m(\Omega)$.

Fact (Schwarz) If $f \in C^2(\Omega)$ then $\frac{\partial^2 f}{\partial x^i x^j} = \frac{\partial^2 f}{\partial x^j x^i}$ and in general for $f \in C\text{^}m(\Omega)$, all partial derivatives of order $\leq m$ are independent of the order of differentiation.

Using higher order derivatives one can analogous to the 1-dimensional case define a Taylor approximation of f.

Fact Let $f \in C^m(\Omega), f: \Omega \to \mathbb{R}, \Omega \in \mathbb{R}, x_0, x_1 \in \Omega$. Then $f(x_1) = f(x_0) + \nabla f(x_0)(x_1 - x_0) + \frac{1}{2}\sum_{i,j=1}^2 \frac{\partial^2 f}{\partial x^i x^j}(x_0)(x_1^i - x_0^i)(x_1^j - x_0^j) + R(f, x_1, x_0)$ where $\lim\limits_{x_1 \to x_0} \frac{R(f, x_1, x_0)}{\|x_1 - x_0\|} \to 0$.

The analog of the second derivative is given by the matrix of partial derivatives of order 2. The matrix is called the Hesse matrix of f.

$$\mathrm{Hess}\, f := \nabla^2 f := \left(\frac{\partial^2 f}{\partial x^i x^j}\right)_{i,j=1\dots n}$$

3.4 The extrema of a function $f: \Omega. \to \mathbb{R}$

Definition A point $x \in \Omega$ is called a critical point if $\nabla f(x) = 0$

Fact If f is differentiable and x_0 is a local extrema of f, then x_0 is a critical point.

Fact Let x_0 be a critical point of f. Then we have

1. x_0 is a local minimum if $\nabla^2 f(x_0)$ is positive definite ($\det \mathrm{Hess}\, f > 0 \wedge \mathrm{tr}\, \mathrm{Hess}\, f > 0$)
2. x_0 is a local maximum if $\nabla^2 f(x_0)$ is negative definite ($\det \mathrm{Hess}\, f > 0 \wedge \mathrm{tr}\, \mathrm{Hess}\, f < 0$)
3. Otherwise x_0 is a saddle point ($\det \mathrm{Hess}\, f < 0$)

To find extrema of f on a region Ω.

1. Find critical points $\Rightarrow \nabla f = 0, x_0$ is a critical point
2. Check the nature of critical points by $\mathrm{Hess}(f)(x_0)$
3. Check the critical points that arise from here

$$H_f(x) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)_{i,j=1,\dots,n} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial 2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(x) \end{pmatrix}_{n \times n}$$

$$\text{Falls 2D: } H_f(x) := \left(\frac{\partial^2 f}{\partial x \partial y}(x)\right)_{i,j=1,\dots,n} = \begin{pmatrix} \frac{\partial^2 f}{\partial x \partial x}(x) & \frac{\partial^2 f}{\partial x \partial y}(x) \\ \frac{\partial^2 f}{\partial y \partial x}(x) & \frac{\partial^2 f}{\partial y \partial y}(x) \end{pmatrix}_{2 \times 2}$$

The Jacobi-Matrix works similar to the Hesse-Matrix, but it only uses the first derivatives.

Fact Let $f: \Omega \to \mathbb{R}$ be continours and differentiable on an open set $\Omega \subset \mathbb{R}^n$. Let $\partial\Omega$ be the boundary of Ω. Then every global extrema of f is either a critical point of f in Ω or a global extramal point of $f|_{\partial x}$.

3.5 Line integral

Let $v: \Omega \to \mathbb{R}^n$ be a vector field and γ a curve with parameterization $\gamma: [a,b] \to \Omega, t \to \gamma(t)$. Then the line integral of v along γ is deinfed as $\int_\gamma v \cdot \overrightarrow{ds} := \int_a^b \langle v(\gamma(t)), \gamma'(t) \rangle \, dt$.

Facts

1. $\int_\gamma v \, ds$ is independent of the parameterization of the path
2. $\int_{\gamma_1+\gamma_2} v \ ds = \int_{\gamma_1} v \ ds + \int_{\gamma_2} v \ ds$
3. $-\int_\gamma v \, ds = -\int_{-\gamma} v \ ds$, where $-\gamma$ is the same path as γ in opposite direction
4. If v is the gradient vector field associated to a function f i.e. $v = df$, then $\int_\gamma v \ ds = f(\gamma(b)) - f(\gamma(a))$, $\gamma: [a,b] \to \Omega$
5. Wir können den Begriff des Wegintegrals auf Wege erweitern, die stückweise C^1 sind. Ein stückweise C^1-Weg ist eine stetige Abbildung $\gamma: [a,b] \to \mathbb{R}^n$ mit einer endlichen Unterteilung des Intervalls $a = c_0 < c_1 < \cdots < c_n = b$ so dass $\gamma|_{[c_i, c_{i+1}]}: [c_i, c_{i+1}] \to \mathbb{R}^n, (i = 0 \dots n-1)$ in C^1 ist. Dann definiert man: $\int_\gamma \lambda := \sum_{i=0}^{n-1} \int_{(\gamma|_{[c_i, c_{i+1}]})} \lambda$

Equivalent once can write everything in terms of 1 forms: $\lambda = \lambda_1 dx^1 + \lambda_2 dx^2 + \cdots + \lambda_n dx^n$ then $\int_\gamma \lambda = \int_a^b \lambda(\gamma(t)) \cdot \gamma'(t) \, dt$

Facts $\lambda: \Omega \to L(\mathbb{R}^n \to \mathbb{R})$ a continuous 1 forms, then the following are equivalent

1. $\exists f \in C^1(\Omega)$ st $df = \lambda$
2. For every two continuous C^1 paths γ_1, γ_2 with the same beginning and end points: $\int_{\gamma_1} \lambda = \int_{\gamma_2} \lambda$
3. For every closed curve γ, $\int_\gamma \lambda = 0$

Definition A vector field $v: \Omega \to \mathbb{R}^n$ is called conservative if $\int_\gamma v \ ds = 0$ for all closed curves γ.

Fact For a simply connected region Ω, we have: v conservative $\Leftrightarrow v = \nabla f$ for some function f.

3.6 div, rot, …

- $\operatorname{div} K := \nabla \cdot K = \frac{\partial K_1}{\partial x} + \frac{\partial K_2}{\partial y} + \frac{\partial K_3}{\partial z}$
- $\operatorname{grad} f := \nabla f = \left(\frac{\partial f}{\partial x^1}(x_0), \dots, \frac{\partial f}{\partial x^n}(x_0)\right)$, in 3D: $\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$, Richtungsableitung: $\nabla f \cdot \vec{r}$

- $\text{rot}\, K := \nabla \times K = \begin{pmatrix} \frac{\partial K_3}{\partial y} - \frac{\partial K_2}{\partial z} \\ \frac{\partial K_1}{\partial z} - \frac{\partial K_3}{\partial x} \\ \frac{\partial K_2}{\partial x} - \frac{\partial K_1}{\partial y} \end{pmatrix}$
- $\text{div}(fK) = \nabla f \cdot K + f \cdot \text{div}\, K$
- $\text{div}(K \times L) = L \cdot \text{rot}\, K - K \cdot \text{rot}\, L$
- $\text{rot}(\text{grad}\, f) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- $\text{div}(\text{rot}\, K) = 0$
- $\text{div}(f \cdot \text{rot}\, K) = \text{grad}\, f \cdot \text{rot}\, K$

3.7 Tangentialebene ausrechnen

Um die Tangentialebene am Graph $\mathcal{G}(t)$ auszurechnen, gibt es drei Möglichkeiten. $T(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(\dots)(x-x_0) + \frac{\partial f}{\partial y}(\dots)(y-y_0) = \mathrm{d}f(x,y)$

- 1. Möglichkeit: $T(x,y)$ ausrechnen
- 2. Möglichkeit: Tangentialvektoren; Tangentialvektoren sind immer: $\begin{pmatrix} \nabla f(x,y) \\ -1 \end{pmatrix}$
- 3. Möglichkeit: Linearkombination

3.8 Potential ausrechnen[1]

Sei $\vec{F} = (6xy + 4z^2, 3x^2 + 3y^2, 8xz)$ ein Vektorfeld. Bestimme das Potential f

- $\frac{\partial f}{\partial x} = 6xy + 4z^2 \to \int (6xy + 4z^2)dx = 3x^2y + 4xz + \overbrace{g(y,z)}^{=y^3+h(z)} = f(x,y,z)$
- $\frac{\partial f}{\partial y} = 3x^2 + 3y^2 \to \frac{\partial f}{\partial y} = 3x^2 + \frac{\partial g}{\partial y}(y,z) \to \frac{\partial g}{\partial y} = 3y^2 \to \int 3y^2\, dy = y^3 + h(z)$
- $\frac{\partial f}{\partial z} = 8xz \to \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(3x^2y + 4xz + y^3 + h(z)\right) = 8xz + \underbrace{g'(z)}_{=\frac{\partial g}{\partial z}} \to g'(z) = 0, \int 0\, dz = c$
- $\to f(x,y,z) = 3x^2y + 4xz + y^3 + c$

3.9 Additional Wisdom

- $\frac{d}{dt} f\big(x(t), y(t)\big) = df\big(x(t), y(t)\big) \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \frac{\partial f}{\partial x}\big(x(t), y(t)\big) \cdot x'(t) + \frac{\partial f}{\partial y}\big(x(t), y(t)\big) \cdot y'(t)$ (chain rule)
- Ein geschlossener Weg in einem konservativen Vektorfeld ist $= 0$, $\int_\gamma \vec{E} \cdot \overrightarrow{ds} = 0$.

4 Integration in $\mathbb{R}^n$

The Riemann integral in $\mathbb{R}^n$ is constructed in an analog way to the case $n = 1$ with Riemann sums over subintervals replaced with sums over "subrectangles", with dx replaced with a n-ddimensional volume element $\mathrm{d}vol_n$ which we denote either by $\mathrm{d}vol_n$ or $d\mu(\vec{x})$.

Fact For a rectangle $Q = [a,b] \times [c,d] \in \mathbb{R}^2$: $\int_Q f\, d\mu = \int_a^b \int_c^d f\, dy\, dx = \int_c^d \int_a^b f\, dx\, dy$.

Fubini $\int_J F(t)dt = \int_J \int_I f(x,y)\, dxdy = \int_I \int_J f(x,y)dydx = \int_{I\times J} f(x,y)\, d(x,y)$

4.1 Substitution in $\mathbb{R}^n$

Let $u, v \in \mathbb{R}^n$ open, $\Phi: u \to v$ bijective with $\det \Phi \neq 0\ \forall \in \tilde{u}$. Then for $f = v \to \mathbb{R}$ continuous we have $\int_v f(\vec{x})d\mu(\vec{x}) = \int_u f\big(\Phi(y)\big)\big|\det\big(d\Phi(y)\big)\big| d\mu(\vec{y})$

Theorem 9.17 $U, V \subset \mathbb{R}$ open, $\Phi: U \to V$ bijective, continuous, differentiable, $\det d\Phi(\vec{y}) \neq 0 \forall \vec{y} \in U$, $f: V \to \mathbb{R}$ continuous. $\int_V f(\vec{x})d\mu(\vec{x}) = \int_{\Phi(U)=V} f\big(\Phi(\vec{y})\big)\, |\det d\Phi(\vec{y})|\, d\mu$. $d\Phi(\vec{y})$ is the Jacobi matrix.

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[1] https://www.youtube.com/watch?v=tslJEOnt9aY

4.2 Green's theorem

Let $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ has a C^1 parameterization. Let $U \subset \Omega$ and $f = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ where $P, Q \in C^1(U)$. Then $\int_\Omega \int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\mu = \int_{\partial\Omega} P\, dx + Q\, dy$

OR

Let $V = (P, Q)$ be a vector field then $\int_{\partial\Omega} v\ ds = \int_\Omega \int \operatorname{rot} v\ d\mu$ where $\operatorname{rot} V = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ and the line integral is taken round the boundary of Ω in counter-clockwise direction.

4.3 Additional Wisdom

- **Parameterintegral**[2]

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\, dt\right) = f\big(x, b(x)\big)b'(x) - f\big(, a(x)\big)a'(x) + \int_{a(x)}^{b(x)} f_x(x,t)\, dt$$

- **Mehrdimensionale Integration** Bei der Koordinatentransformation das "r" (o.ä.) nicht vergessen

[2] Differentiation under the integral sign