Vectors

A vector is a certain displacement with a certain direction, but its origin is not defined. Two vectors are equal, if they have the same magnitude and direction.

1 Operations

Addition and subtraction of vectors is done by applying the respective operation on each component (e.g. $\begin{pmatrix} a_x \\ a_y \end{pmatrix} + \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix}$). Commutative and associative law apply $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ | $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$. The zero vector $\vec{o}$ is a vector with magnitude 0 and undefined direction, you can get it by e.g. $\vec{a} - \vec{a} = \vec{o}$. Scalar multiplicity works as following: $n * \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} n * a_x \\ n * a_y \end{pmatrix}$ (associative and distributive laws apply).

2 Position Vectors

If a point should be described, a position vector $\vec{p}$ is used, e.g. point $P(3|2|8)$ is described as $\vec{p} = \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix}$. The vector between point $A(2|5|9)$ and point $B(3|8|15)$ is $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$.

3 Magnitude

The magnitude of a vector is noted as $|\vec{a}|$ and calculated like this: $\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$. A vector with length (magnitude) 1 is called unit vector and a vector with the same direction as $\vec{a}$ and length 1 is denoted as $\overrightarrow{a_E}$ and calculated by $\overrightarrow{a_E} = \frac{1}{|\vec{a}|} * \vec{a}$. For magnitude the following theorem applies: $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

4 Linear Dependency

A vector $\vec{p}$ is a linear combination of the vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \dots \overrightarrow{v_n}$, if there are $r_1, r_2, \dots r_n$ such that $\vec{p} = r_1 * \overrightarrow{v_1} + r_2 * \overrightarrow{v_2} + r_n * \overrightarrow{v_n}$. If such a combination exists, the set of vectors is called linearly dependent, if not it is called linearly independent, which is only the case if:

$$(A)\overrightarrow{v_1} \neq 0, \overrightarrow{v_2} \neq 0, \dots \overrightarrow{v_n} \neq 0$$
$$and$$
$$(B) r_1 * \overrightarrow{v_1} + r_2 * \overrightarrow{v_2} + r_n * \overrightarrow{v_n} = 0 \; which \; implies \; r_1 = r_2 = \cdots = r_n = 0$$

Any vector $\vec{p}$ can be constituted in 2D of two and in 3D of three, linearly independent vectors (called base vectors) as a unique linear caombination of $\vec{p} = r_1 * \overrightarrow{v_1} + r_2 * \overrightarrow{v_2} (+ r_3 * \overrightarrow{v_3})$. Typical base vectors are in 2D $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and in 3D $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

5 Dot Product

The dot product (scalar/inner product) is calculated as $\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} * \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x * b_x + a_y * b_y + a_z * b_z$.If this product equals zero, the two vectors are perpendicular. The angle between two vectors can be calculated as follows: $\cos(\alpha) = \frac{\vec{a} * \vec{b}}{|\vec{a}| * |\vec{b}|}$.

6 Lines

Any point X with position vector $\vec{x}$ on a line can be calculated by adding a scaled vector to a certain point $\vec{a}$: $\vec{a} = \vec{x} + t * \overrightarrow{AB}$. E.g. the equation of the line passing through point $A(2|5|9)$ and point $B(3|8|15)$ is calculated as follows:

$$\vec{a} = \begin{pmatrix} 2 \\ 5 \\ 9 \end{pmatrix}$$

$$\overrightarrow{AB} = \vec{b} - \vec{a} = \begin{pmatrix} 3 \\ 8 \\ 15 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

$$g: \vec{x} = \begin{pmatrix} 2 \\ 5 \\ 9 \end{pmatrix} + t * \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

6.1 Position Relations

To check how lines $g: \vec{x} = \vec{a} + s * \vec{u}$ and $h: \vec{x} = \vec{b} + t * \vec{v}$ are related, you can do the following:

1. Check whether the direction vectors are linearly dependent. If yes, they are parallel or identical (continue with 2a). If no, they are intersecting or skew (continue with 2b).

2a. Check whether the lines have a common point. If they have a common point (you can try with the point described by the position vector of one of the lines), they are identical. If they don't, they're parallel.

2b. Check whether the lines have a common point by equating the two line equations (make sure you use s and t, not t twice...). If you get a solution, the lines intersect, if you don't, they're skew.

- Linear Dependency, Parallel, Identical
  Find a number r such that $\vec{u} = r * \vec{v}$
- Intersection Point

$$a_x + s * u_x = b_x + t * v_x$$
$$a_y + s * u_y = b_y + t * v_y$$
$$a_z + s * u_z = b_z + t * v_z$$

  solve equations 1 and 2 for s and t and plug in to equation 3 to check whether equation 3 is stratified. If it is, you can calculate the intersection point plugging in s and t into g and h.

7 Planes

A plane is uniquely defined by

- Three distinct points
- A line and a point
- Two intersecting lines
- A direction perpendicular to the plane and a point on the plane

A plane can be (uniquely) defined by three different forms:

- Vector equation $E: \vec{x} = \vec{a} + t * \vec{u} + s * \vec{v}$
- Normal vector equation $E: (\vec{x} - \vec{p}) * \vec{n} = 0$
- Cartesian equation $E: a * x + b * y + c * z = d$

7.1 Obtaining Equations

7.1.1 Vector Equation

To get the vector equations from three, defined points A, B, C the following method is applied: $E: \vec{x} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} + t * \begin{pmatrix} b_x - a_x \\ b_y - a_y \\ b_z - a_z \end{pmatrix} + s * \begin{pmatrix} c_x - a_x \\ c_y - a_y \\ c_z - a_z \end{pmatrix}$. To check whether a point P is on the plane, t and s have to be found such that:

$$p_x = a_x + t * b_x + s * c_x$$
$$p_y = a_y + t * b_y + s * c_y$$
$$p_z = a_z + t * b_z + s * c_z$$

7.1.2 Normal Equation

To find a vector perpendicular to $\vec{u}$ and $\vec{v}$ called $\vec{n}$ is obtained by:$u_x * n_x + u_y * n_y + u_z * n_z = 0 \; and \; v_x * n_x + v_y * n_y + v_z * n_z = 0$. If you solve this system with your calculator, you get a solution which contains "@". Plug in any number for @ (this changes the magnitude of the normal vector).

The equation reads as follows: $E: \left(\vec{x} - \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \right) * \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = 0$ whereas $\vec{a}$ is a point on the plane.

7.1.3 Cartesian Equation

By expanding $E: \left(\vec{x} - \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \right) * \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = 0$ and replacing $\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ by $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$a unique equation is obtained:

$$\left(\begin{pmatrix} x_x \\ x_y \\ x_z \end{pmatrix} - \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \right) * \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$
$$\begin{pmatrix} x_x \\ x_y \\ x_z \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\begin{pmatrix} x_x \\ x_y \\ x_z \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$
$$a * x_x + b * x_y + c * x_z = a * p_x + b * p_y + c * p_z$$
$$a * x_x + b * x_y + c * x_z = d$$

7.2 Transforming Equations

7.2.1 Vector → Normal

Solve these two equations for $\vec{n}$.

$$\vec{v} * \vec{n} = 0$$
$$\vec{u} * \vec{n} = 0$$

And set $\vec{a} = \vec{p}$

7.2.2 Normal → Vector

Set $\vec{p} = \vec{a}$ and find two linearly independent and perpendicular to $\vec{n}$ vectors (not unique) such that

$$\vec{v} * \vec{n} = 0$$
$$\vec{u} * \vec{n} = 0$$

7.2.3 Vector → Cartesian

Solve this system of equations as described:

- solve the first equation for t
- plug t into the second and third equation
- solve the second equation for s
- plug s into the third equation
- simplify

If it helps, you can also change the order of the equations.

$$x = a_x + t * u_x + s * v_x$$
$$y = a_y + t * u_y + s * v_y$$
$$z = a_z + t * u_z + s * v_z$$

7.2.4 Cartesian → Vector

$$a * x_x + b * x_y + c * x_z = d$$

First obtain three points on the plane by plugging in 0 for two variables and solving for the third variable to obtain $A, B \text{ and } C$. This will give you something like $A(a_x|0|0), B(0|b_y|0) \text{ and } C(0|0|c_z)$.

$$E: \vec{x} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} + t * \begin{pmatrix} b_x - a_x \\ b_y - a_y \\ b_z - a_z \end{pmatrix} + s * \begin{pmatrix} c_x - a_x \\ c_y - a_y \\ c_z - a_z \end{pmatrix}$$

7.2.5 Normal → Cartesian

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{n}$$
$$d = \vec{n} * \vec{p}$$

7.2.6 Cartesian → Normal

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

And find a point on the plane for $\vec{p}$.